THE ALEXANDER- AND JONES-INVARIANTS AND THE BURAU MODULE

FLORIN CONSTANTINESCU AND MIRKO LÜDDE

Abstract. From the braid-valued Burau module over the braid group we construct the Yang-Baxter matrices yielding the Alexander- and the Jones knot invariants. This generalises an observation of V. F. R. Jones.

1. Introduction

It has been known for long that the Burau representation of Artin's braid group B_n can be used to construct the Alexander polynomial invariant of knots. There are at least two ways to accomplish this. The topological approach uses the fact that the first relative homology of the cyclic covering of the knot's complement has a presentation as a $\mathbb{Z}[t,t^-]$ module determined by the Burau matrices, cf. [BZ85]. Another approach, the one which is generalised in this article, constructs a solution of the Yang-Baxter equation starting with the Burau representation. This proceeds by extending the 3-dimensional Burau representation of B_3 to the 2^3-dimensional Grassman algebra of the representation space. The Yang-Baxter matrix can then be turned into the Alexander invariant, e. g. by a state model on knot diagrams, cf. [Jon91, Kau91]. This idea goes back to Jones and has been investigated in [Kau91, KS92].

In this article, following a proposal of [Con95], we will obtain the simplest Yang-Baxter solution associated to the deformation $U_t(\mathrm{sl}(2))$, and therefore the Jones invariant. We use a generalisation of the Burau representation that we called the "braid-valued Burau representation" in [CL92, Lüd92]. By this term we mean the module obtained as the relative augmentation ideal of the free group F_n of rank n in the integral group ring $\mathbb{Z}[B_n \ltimes F_n]$. Tensor products of these modules can be suitably reduced: an antisymmetrisation of tensor products yields the Grassman algebra carrying the "classical" Burau representation and therefore the Alexander invariant. By similar but different relations the Yang-Baxter matrix for the Jones invariant is obtained.

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2. The Burau representation

The definitions and facts on the braid group used here can be found e.g. in [Bir74, BZ85], if not stated otherwise. The braid group B_n on n strings (over the Euclidean plane) is the group generated by the set $\{\tau_i; i \in \{1,\ldots,n-1\}\}$ according to the relations of Artin, $\tau_i\tau_j = \tau_j\tau_i$, if $\mathrm{abs}(i-j) \geq 2$, $\tau_i\tau_{1+i}\tau_i = \tau_{1+i}\tau_i\tau_{1+i}$.

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The braid group B_n faithfully acts onto the free group $F_n :=< f_1, \dots, f_n >$ of rank n. There is a monomorphism $\psi \in \mathrm{Hom}(B_n, \mathrm{Aut}(F_n))$ (where we let the automorphisms act from the right) defined by

$$\psi(\tau_i) : f_j \mapsto \begin{cases} f_i f_{i+1} f_i^{-1}, & j = i, \\ f_i, & j = i+1, \\ f_j, & j \notin \{i, i+1\} \end{cases} .$$

We will only need the fact that ψ is a (anti-)homomorphism, which can be checked by computation.

Now we can define the semidirect product $B_n F_n := B_n \ltimes_\psi F_n$ as the set $B_n \times F_n$ with multiplication $(\alpha, f)(\beta, g) := (\alpha\beta, (\psi(\beta))(f)g)$, which we will write simply as $\alpha f \beta g = \alpha\beta\beta(f)g$.

There are several approaches to the classical Burau representation. Following W. Magnus, cf. [Jon91, Mag74], one may investigate the B_n action on abelianised groups $U/[U,U]$ for suitable $U \le F_n$. A different method proceeds by using the Fox-derivative on the free group, cf. [Bir74]. A topological construction via a flat connexion on a homology bundle is obtained as a particular case in [Ati90, Law90]. The approach we are using presumably had not been noted in the literature until [CL92, Lüd92]. For an algebraic derivation of all these methods and their generalisations see [Lüd95].

1 (Braid-valued Burau module)**.** *The map*

$$\tau_i \mapsto \tau_i \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & (1 - f_i f_{i+1} f_i^{-1}) & f_i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}$$

uniquely extends to a monomorphism $B_n \to \mathrm{GL}(n, \mathbb{Z}[B_n F_n])$.

Proof. The relative augmentation ideal $\mathrm{Lin}_{\mathbb{Z}B_nF_n}\{(f_i - 1); i \in \{1, \dots, n\}\}$ of the free group in the integral group ring $\mathbb{Z}[B_n F_n]$ is free of rank n as a left $B_n F_n$ module over the set $\{s_i := (f_i - 1); i \in \{1, \dots, n\}\}$. As an ideal, by multiplication from the right, it is a module over B_n, $(f_j - 1) \mapsto (f_j - 1)\tau_i = \tau_i(\tau_i(f_j) - 1)$. The element $\tau_i(f_j) := \psi(\tau_i)(f_j)$ is determined by Artin's action, so we obtain the equation $s_j \tau_i = \tau_i \begin{cases} (1 - f_i f_{i+1} f_i^{-1}) s_i + f_i s_{i+1}, & j = i \\ s_i, & j = i+1 \\ s_j, & j \notin \{i, i+1\} \end{cases}$. This action is faithful, since the matrix representative of $\alpha \in B_n$ has the form αS, with S an n by n matrix over the ring $\mathbb{Z}F_n \hookrightarrow \mathbb{Z}B_nF_n$. □

Example 1 (Burau module)**.** *The ring homomorphism* $\mathbb{Z}[B_n F_n] \to \mathbb{Z}[t, t^-]$, *with an indeterminate* t, *defined by* $\tau_i \mapsto 1$, $f_j \mapsto t$, *applied to the braid-valued Burau matrices, yields the representation*

$$\tau_i \mapsto \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix} .$$

3. Construction of Yang-Baxter matrices

Having at hand the Burau representation, we recall Jones' construction of a Yang-Baxter matrix from it. This prepares us for the general procedure.

2 (Alexander invariant from Burau module)**.** *Let $R := \mathbb{Z}[t, t^-]$ be the ring of Laurent polynomials in an indeterminate t. Let $\rho \in \mathrm{Hom}(B_3, \mathrm{Aut}(R^3))$ be a Burau representation, given by the matrices* $\rho_1 := \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\rho_2 := \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, *where* $B := \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$. *Then the natural extension of ρ to the exterior algebra $\Lambda(R^3)$ (where the ρ_i act as algebra homomorphisms) is isomorphic to a Yang-Baxter representation $\Upsilon \in \mathrm{Aut}(V \otimes V)$ with a 2-dimensional free R module V.*

Proof. The statement and proof are taken from [Kau91], sect. I.13, pp. 208. Let R^3 as an R module have the basis $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$. The representation ρ acts by multiplication from the right onto these row vectors. Let V be the free left R module with basis e_1, e_2. Then we define an isomorphism of free left R modules $\phi \in \mathrm{Hom}(\Lambda(R^3), V \otimes V \otimes V)$ by sending $(1, v_1, v_2, v_1 \wedge v_2, v_3, v_1 \wedge v_3, v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_3)$ to $(e_1e_1e_1, e_1e_2e_1, e_2e_1e_1, e_2e_2e_1, e_1e_1e_2, e_1e_2e_2, e_2e_1e_2, e_2e_2e_2)$. Computing $\sigma_i = \phi \circ \rho_i \circ \phi^{-1}$ we find, $\sigma_1 = \Upsilon \otimes 1$ and $\sigma_2 = 1 \otimes \Upsilon$ with the matrix

$$\Upsilon := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}$$

(where the rows correspond to the coefficients of the images $\Upsilon(e_i \otimes e_j)$ in the ordered basis e_1e_1, e_1e_2, e_2e_1, e_2e_2.) This matrix satisfies the (permuting form of the) Yang-Baxter equation, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, as a consequence of the braid relations obeyed by ρ_1 and ρ_2. □

A rescaling and a change of basis transforms the matrix into the one that via a state model on knot diagrams yields the Alexander invariant, as described in [Kau91], sect. I.12, pp. 174.

We want to apply a similar technique to the braid-valued Burau module. By Artin's combed normal form for the braid group, the group B_nF_n can be imbedded into B_{1+n}. These imbeddings can be iterated to build the groups $B_{n,j} := B_n \ltimes F^{(1)} \ltimes \cdots \ltimes F^{(j)}$, where $F^{(j)} := F_{n+j-1}$. We need to know two facts on this imbedding. The first is that the generators of B_n are mapped as $\tau_i^{(n)} \mapsto \tau_i^{(1+n)}$ for $i \in \{1, \ldots, n-1\}$, where superscripts indicate the respective groups. The second is, in which way the image of the free group $F^{(j)} :=< f_1^{(j)}, \ldots, f_{n+j-1}^{(j)} >$ in $B_{n+j} \hookrightarrow \mathrm{Aut}(F^{(l)})$ acts onto the generators of $F^{(l)}$ for $l > j$. This action is given by ($\epsilon \in \{-1, 1\}$)

$$f_i^{(j)\epsilon}(f_k^{(l)}) = \begin{cases} f_k^{(l)}, & k < i \text{ or } n+j-1 < k \\ \mathrm{Ad}((f_i^{(l)} f_{n+j-1}^{(l)})^{\epsilon})(f_k^{(l)}), & k \in \{i, n+j-1\} \\ \mathrm{Ad}([f_{n+j-1}^{(l)\epsilon}, f_i^{(l)\epsilon}]^{-\epsilon})(f_k^{(l)}), & i < k < n+j-1 \end{cases},$$

where $k < n+l-1$, $i < n+j-1$, $\mathrm{Ad}(x)(y) := xyx^-$, $[x, y] := xyx^-y^-$. These equations are equivalent to the relations for the generators of the pure braid group. We consider the groups $B_{3,j}$ for $j \in \{1, 2, 3\}$. Let $I^{(j)} := \mathrm{Lin}_{\mathbb{Z}B_{3,j}}\{s_i^{(j)} := (f_i^{(j)} - 1)\}$

be the relative augmentation ideal of $F^{(j)}$ in the ring $\mathbb{Z}B_{3,j}$. Furthermore, let a left $B_{3,3}$ right B_3 bimodule be defined as the sum of tensor products

$$M := \mathbb{Z}[B_{3,3}]\mathbf{1} \oplus I^{(3)} \oplus I^{(3)} \otimes_{B_{3,2}} I^{(2)} \oplus I^{(3)} \otimes_{B_{3,2}} I^{(2)} \otimes_{B_{3,1}} I^{(1)}.$$

Define a right $B_{3,3}$ module structure on the ring $\mathbb{Z}[t,t^-]$ of Laurent polynomials by mapping the generators of $B_{3,3}$ as $\tau_i \mapsto 1$, for $i \in \{1,2\}$ and $f_k^{(j)} \mapsto t$. Guided by Jones' construction of the representation on the Grassman algebra, we construct quotients of rank 2^3 of M.

3 (Invariants from braid-valued Burau module)**.** *The representation of B_3 defined by the Yang-Baxter matrix*

$$R := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

as well as the representation by the previously defined matrix Υ can be obtained as suitable quotients from the tensor product $\mathbb{Z}[t,t^-] \otimes_{B_{3,3}} M$ regarded as left $\mathbb{Z}[t,t^-]$ right B_3 bimodule.

Proof. We compute the right action of τ_1 induced by Artin's automorphisms on some particular basis elements of M. For notational convenience we drop the tensor product symbol and set $a_i^{(j)} := (1 - f_i^{(j)} f_{1+i}^{(j)} f_i^{(j)-})$. We get $\mathbf{1}\tau_1 = \tau_1\mathbf{1}$, $s_1^{(3)}\tau_1 = \tau_1(a_1^{(3)}s_1^{(3)} + f_1^{(3)}s_2^{(3)})$, $s_2^{(3)}\tau_1 = \tau_1 s_1^{(3)}$, $s_3^{(3)}\tau_1 = \tau_1 s_3^{(3)}$, $s_1^{(3)}s_2^{(2)}\tau_1 = \tau_1(a_1^{(3)}s_1^{(3)} + f_1^{(3)}s_2^{(3)})s_1^{(2)}$, $s_1^{(3)}s_3^{(2)}\tau_1 = \tau_1(a_1^{(3)}s_1^{(3)} + f_1^{(3)}s_2^{(3)})s_3^{(2)}$, $s_2^{(3)}s_3^{(2)}\tau_1 = \tau_1 s_1^{(3)}s_3^{(2)}$ and finally, $s_1^{(3)}s_2^{(2)}s_3^{(1)}\tau_1 = \tau_1(a_1^{(3)}s_1^{(3)} + f_1^{(3)}s_2^{(3)})s_1^{(2)}s_3^{(1)}$. Similarly, the action of the second generator τ_2 is as follows: $\mathbf{1}\tau_2 = \tau_2\mathbf{1}$, $s_1^{(3)}\tau_2 = \tau_2 s_1^{(3)}$, $s_2^{(3)}\tau_2 = \tau_2(a_2^{(3)}s_2^{(3)} + f_2^{(3)}s_3^{(3)})$, $s_3^{(3)}\tau_2 = \tau_2 s_2^{(3)}$, $s_1^{(3)}s_2^{(2)}\tau_2 = \tau_2(a_2^{(2)}s_1^{(3)}s_2^{(2)} + f_2^{(2)}s_1^{(3)}s_3^{(2)})$, $s_1^{(3)}s_3^{(2)}\tau_2 = \tau_2 s_1^{(3)}s_2^{(2)}$, $s_2^{(3)}s_3^{(2)}\tau_2 = \tau_2(a_2^{(3)}s_2^{(3)} + f_2^{(3)}s_3^{(3)})s_2^{(2)}$, $s_1^{(3)}s_2^{(2)}s_3^{(1)}\tau_2 = \tau_2(a_2^{(2)}s_1^{(3)}s_2^{(2)}s_2^{(1)} + f_2^{(2)}s_1^{(3)}s_3^{(2)}s_2^{(1)})$. We now pass to the quotient of $\mathbb{Z}[t,t^-] \otimes_{B_{3,3}} M$ as a left $\mathbb{Z}[t,t^-]$ module by imposing the relations (note the similarity with the relations of a Grassman algebra)

$$s_i^{(3)}s_j^{(2)} = \begin{cases} 0, & i = j \\ t^- s_j^{(3)}s_i^{(2)}, & i > j \end{cases},$$

$$s_i^{(3)}s_j^{(2)}s_k^{(1)} = \begin{cases} 0, & \mathrm{card}\{i,j,k\} < 3 \\ t^- s_j^{(3)}s_i^{(2)}s_k^{(1)}, & i > j \\ t^- s_i^{(3)}s_k^{(2)}s_j^{(1)}, & j > k \end{cases}.$$

This quotient Q is a free left $\mathbb{Z}[t,t^-]$ module with basis given by the 2^3 elements $(1, s_1, s_2, s_1s_2, s_3, s_1s_3, s_2s_3, s_1s_2s_3)$. We obtain an induced action of τ_1 and τ_2 on Q, $\mathbf{1}\tau_1 = \mathbf{1}$, $s_1^{(3)}\tau_1 = (1-t)s_1^{(3)} + ts_2^{(3)}$, $s_2^{(3)}\tau_1 = s_1^{(3)}$, $s_3^{(3)}\tau_1 = s_3^{(3)}$, $s_1^{(3)}s_2^{(2)}\tau_1 = s_1^{(3)}s_2^{(2)}$, $s_1^{(3)}s_3^{(2)}\tau_1 = ((1-t)s_1^{(3)} + ts_2^{(3)})s_3^{(2)}$, $s_2^{(3)}s_3^{(2)}\tau_1 = s_1^{(3)}s_3^{(2)}$, $s_1^{(3)}s_2^{(2)}s_3^{(1)}\tau_1 = s_1^{(3)}s_2^{(2)}s_3^{(1)}$ and similarly for the action of τ_2. In order to obtain a Yang-Baxter representation of B_3, consider the free left $\mathbb{Z}[t,t^-]$ module $V \otimes V \otimes V$ of rank 2^3 with $V := \mathbb{Z}[t,t^-]^2$ and define an isomorphism $Q \to V \otimes V \otimes V$ by sending the ordered basis above to $(e_1e_1e_1, e_1e_2e_1, e_2e_1e_1, e_2e_2e_1, e_1e_1e_2, e_1e_2e_2, e_2e_1e_2, e_2e_2e_2)$. On $V \otimes V \otimes V$ the braid generators τ_1 and τ_2 are then found to act by matrices $R \otimes 1$ and $1 \otimes R$, respectively.

Finally we notice that a similar construction, where we impose the exact Grassman relations, leads to the matrix Υ precisely as in the previous lemma. $\square$

By a rescaling and a change of basis in $V \otimes V$, the Yang-Baxter matrix R is found to be the universal R-matrix of $U_t(\mathrm{sl}(2))$ in its fundamental representation, cf. [CP94], ex. 6.4.12, pp. 205. So either by a state model, see [Kau91], sect. I.11, pp. 161, or by Turaev's theorem, see [CP94], sect. 15.2, pp. 504, the Jones polynomial can be obtained.

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Institut für Angewandte Mathematik, Johann Wolfgang Goethe Universität, Robert Mayer Str. 6-10, D-60054 Frankfurt am Main, Germany, email: 'constantinescu@mathematik.uni-frankfurt.d400.de'

Institut für Reine Mathematik, Humboldt Universität zu Berlin, Ziegelstrasse 13a, D-10099 Berlin, Germany, email:'luedde@mathematik.hu-berlin.de'